



# Random Graph Models with Non-Independent Edges

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## Abstract

Random graph models play an important role in describing networks with random structural features. The most classical model with the largest number of existing results is the Erdős-Rényi random graph, in which the edges are chosen interdependently at random, with the same probability. In many real-life situations, however, the independence assumption is not realistic. We consider random graphs in which the edges are allowed to be dependent. In our model the edge dependence is quite general, we call it  $p$ -robust random graph. Our main result is that for any monotone graph property, the  $p$ -robust random graph has at least as high probability to have the property as an Erdős-Rényi random graph with edge probability  $p$ . This is very useful, as it allows the adaptation of many results from Erdős-Rényi random graphs to a non-independent setting, as lower bounds.

Random networks occur in many practical scenarios. Some examples are wireless ad-hoc networks, various social networks, the web graph describing the World Wide Web, and a multitude of others. Random graph models are often used to describe and analyze such networks. The oldest and most researched random graph model is the Erdős-Rényi random graph  $G_{n,p}$ . This denotes a random graph on  $n$  nodes, such that each edge is added with probability  $p$ , and it is done independently for each edge. A large number of deep results are available about such random graphs, see expositions in the books [1-4]. Below we list some examples. They are asymptotic results, and for simplicity we ignore rounding issues (i.e., an asymptotic formula may provide a non-integer value for a parameter which is defined as integer for finite graphs).

The size of a maximum clique in  $G_{n,p}$  is asymptotically  $2 \log_{1/p} n$ .

If  $G_{n,p}$  has average degree  $d$ , then its maximum independent set has asymptotic size  $(2n \ln d) / d$ .

The chromatic number of  $G_{n,p}$  is asymptotically  $\frac{n}{\log_b n}$ , where  $b = 1-p$ .

The size of a minimum dominating set in  $G_{n,p}$  is asymptotically  $\frac{n}{\log_b n}$ , where  $b = \frac{1}{1-p}$ .

The length of the longest cycle in  $G_{n,p}$ , when the graph has a constant average degree  $d$ , is asymptotically  $n(1 - de^{-d})$ .

The diameter of  $G_{n,p}$  is asymptotically  $\frac{\log n}{\log(np)}$ , when  $np \rightarrow \infty$ . (If the graph is not connected, then the diameter is defined as the largest diameter of its connected components.)

If  $G_{n,p}$  has average degree  $d$ , then the number of nodes of degree  $k$  in the graph is asymptotically  $\frac{d^k e^{-d}}{k!} n$ .

However, the requirement that the edges are independent is often a severe restriction in modelling real-life networks. Therefore, numerous attempts have been made to develop models with various dependencies among the edges, see a survey in [2]. Here we consider a general form of edge dependency. We call a random graph with this type of dependency a  $p$ -robust random graph.

## Definition 1 ( $p$ -robust random graph)

A random graph on  $n$  vertices is called  $p$ -robust, if every edge is present with probability at least  $p$ , regardless of the status (present or not) of other edges. Such a random graph is denoted by  $\tilde{G}_{n,p}$ .

Note that  $p$ -robustness does not imply independence. It allows that the existence probability of an edge may depend on other edges, possibly in a complicated way, it only requires that the

probability never drops below  $p$ . Let us show some examples of  $p$ -robust random graphs.

1. First note that the classical Erdős-Rényi random graph  $G_{n,p}$  is a special case of our model, since our model also allows taking all edges independently with probability  $p$ .
2. However, we can also allow (possibly messy) dependencies. For example, let  $P(e)$  denote the probability that a given edge  $e$  is present in the graph, and let us condition on  $k$ , the number of other edges in the whole graph. Let  $P_e(k)$  denote the probability that there are  $k$  edges in the graph, other than  $e$ . For any fixed  $k$ , set  $p(e|k) = 1 - (k+1)/n^2$ ; let this be the probability that edge  $e$  exists, given that there are  $k$  other edges in the graph. Using that the total number of edges cannot be more than  $n(n-1)/2$ , we have that  $k \leq n(n-1)/2 - 1$  always holds.

Therefore,  $P(e|k) \geq 1 - \frac{n(n-1)}{2n^2} = 1 - \frac{n-1}{2n} \geq \frac{1}{2}$ , for any  $k$ , implying  $p(e) = \sum_{k=0}^{n(n-1)/2-1} P(e|k)P_e(k) \geq 1/2$ . Thus, with  $p=1/2$ , this random graph is  $p$ -robust. At the same time, the edges are not independent, since the probability that  $e$  is present depends on how many other edges are present.

3. For a given edge  $e$ , let  $r(e)$  denote the number of edges that are adjacent with  $e$  (not includinge itself). If  $e$  does not exist, then let  $r(e) = 0$ . Let the conditional probability that edge exists, given that it has  $k$  adjacent edges, be  $P(e \setminus r(e) = k) = \frac{1}{2} - \frac{1}{k+5}$ . Note that the possible range of  $k$  is  $0 \leq k \leq 2(n-2)$ . Then we have  $P(e \setminus r(e) = k) \geq \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$ . This implies  $P(e) = \sum_{k=0}^{2(n-2)} P(e \setminus r(e) = k)P(r(e) = k) \geq \frac{3}{10} \sum_{k=0}^{2(n-2)} P(r(e) = k) = \frac{3}{10}$ . Thus, with  $P = \frac{3}{10}$ , this random graph is  $p$ -robust. At the same time, the edges are not independent, since the probability that  $e$  is present is influenced by the number of adjacent edges.
4. Consider the model described above in 3, but with the additional condition that each potential edge  $e$  has at least 3 adjacent edges, whether or not  $e$  is in the graph. What can we say about this conditional random graph? The same derivation as in 3, but with  $k \geq 3$ , gives us that the new random graph will remain  $p$ -robust, but now with  $p = 3/8$ .

If we have a random graph like the examples 2,3,4 above (and many possible others with dependent edges), then how can we estimate some parameter of the random graph, like the size of the maximum clique? We show that at least for so called monotone properties we can use the existing results about Erdős-Rényi random graphs as lower bounds. Let  $Q$  be a set of graphs. We use it to represent a graph property: a graph  $G$  has property  $Q$  if and only

if  $G \in Q$ . Therefore, we identify the property with  $Q$ . We are going to consider monotone graph properties, as defined below.

### Definition 2 (Monotone Graph Property)

A graph property  $Q$  is called monotone, if it is closed with respect to adding new edges. That is, if  $G \in Q$  and  $G \subseteq G'$ , then  $G' \in Q$ .

Note that many important graph properties are monotone. Examples: having a clique of size at least  $k$ , having a Hamiltonian circuit, having  $k$  disjoint spanning trees, having chromatic number at least  $k$ , having diameter at most  $k$ , having a dominating set of size at most  $k$ , having a matching of size at least  $k$ , and numerous others. In fact, almost all interesting graph properties have a monotone version. Our result is that for any monotone graph property, and for any  $n, p$ , it always holds that  $\tilde{G}_{n,p}$  is more likely to have the property than  $G_{n,p}$  (or at least as likely). This is very useful, as it allows the application of the rich treasury of results on Erdős-Rényi random graphs to the non-independent setting, as lower bounds on the probability of having a monotone property. Below we state and prove our general result.

**Theorem 1** Let  $Q$  be any monotone graph property. Then the following holds:

$$\Pr(G_{n,p} \in Q) \leq \Pr(\tilde{G}_{n,p} \in Q)$$

**Proof:** We are going to generate  $\tilde{G}_{n,p}$  as the union of two random graphs,  $G_{n,p}$  and  $G_2$ , both on the same vertex set  $V$ .  $G_{n,p}$  is the usual Erdős-Rényi random graph,  $G_2$  will be defined later. The union  $G_{n,p} \cup G_2$  is meant with the understanding that if the same edge occurs in both graphs, then we merge them into a single edge. We plan to choose the edge probabilities in  $G_2$ , such that  $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$ , where the " $\sim$ " relation between random graphs means that they have the same distribution, i.e., they are statistically indistinguishable. If this can be accomplished, then the claim will directly follow, since then a random graph distributed as  $\tilde{G}_{n,p}$  can be obtained by adding edges to  $G_{n,p}$ , which cannot destroy a monotone property, once  $G_{n,p}$  has it. This will imply the claim.

We introduce some notations. Let  $e_1, \dots, e_m$  denote the (potential) edges. For every  $i$ , let  $h_i$  be the indicator of the event that the edge  $e_i$  is included in  $\tilde{G}_{n,p}$ . Further, let us use the abbreviation  $h^m = (h_1, \dots, h_m)$ . For any  $a = (a_1, \dots, a_m) \in \{0,1\}^m$ , the event  $\{h^m = a\}$  means that  $\tilde{G}_{n,p}$  takes a realization in which edge  $e_i$  is included if and only  $a_i = 1$ . Similarly,  $\{h_i^m = a_i^m\}$  means  $\{h_i = a_i, \dots, h_m = a_m\}$ . We also use the abbreviation  $a^m = (a_1, \dots, a_m)$ . Now let us generate the random graphs  $G_{n,p}$  and  $G_2$ , as follows.

Step 1: Let  $i = m$ .

Step 2: If  $i = m$ , then let  $q_m = \Pr(h_m = 1)$ . If  $i < m$ , then set  $q_i = \Pr(h_i = 1 | h_{i+1}^m = a_{i+1}^m)$ , where  $a_{i+1}^m$  indicates the already generated edges of  $G_{n,p} \cup G_2$ .

$$\text{Step 3: Compute } p_i' = \frac{p(1-q_i)}{1-p}. \tag{1}$$

Step 4: Put  $e_i$  into  $G_{n,p}$  with probability  $p$ , and put  $e_i$  into  $G_2$  with probability  $q_i - p_i'$ .

Step 5: If  $i > 1$ , then decrease  $i$  by one, and go to Step 2: else HALT.

First note that the value  $q_i - p_i'$  in Step 4 can indeed be used as a probability. Clearly,  $q_i - p_i' \leq 1$

holds, as  $q_i$  is a probability and  $p_i' \geq 0$ . To show  $q_i - p_i' \geq 0$ , observe that  $p_i' \frac{p(1-q_i)}{1-p} \leq q_i$ , since the inequality can be rearranged into  $p(1-q_i) \leq q_i(1-p)$ , which simplifies to  $p \leq q_i$ . The latter is indeed true, due to  $q_i = \Pr(h_i = 1 | h_{i+1}^m = a_{i+1}^m) \geq p$  which follows from the  $p$ -robust property. Next, we show that the algorithm generates the random graphs  $G_{n,p}$  and  $G_2$  in a way that they satisfy  $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$ . We prove it by induction, starting from  $i = m$  and progressing downward to  $i = 1$ . For any  $i$ , let  $G_{n,p}^i, G_2^i$  denote the already generated parts of  $G_{n,p}, G_2$  respectively, after executing Step 4  $m-i+1$  times, so they can only contain edges with index  $\geq i$ . Further, let  $G_{n,p}^i$  be the subgraph of  $G_{n,p}$  in which we only keep the edges with index  $\geq i$ , that is,  $G_{n,p}^i = \tilde{G}_{n,p} - \{e_{i-1}, \dots, e_1\}$ .

The inductive proof will show that  $G_{n,p}^i \cup G_2^i \sim \tilde{G}_{n,p}^i$  holds for every  $i$ . At the end of the induction, having reached  $i = 1$ , we are going to get  $G_{n,p}^1 \cup G_2^1 \sim \tilde{G}_{n,p}^1$ , which is the same as  $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$ .

Let us consider first the base case  $i = m$ . Then we have  $\Pr(e_m \in G_{n,p}) = \Pr(e_m \in G_{n,p}^m) = p$  by Step 4. Then in Step 4, edge  $e_m$  is put into  $G_2$  with probability  $q_m - p_m'$ , yielding  $\Pr(e_m \in G_2^m) = q_m - p_m'$ . Now observe that the formula (1) is chosen such that  $p_i'$  is precisely the solution of the equation

$$p + q_i - p_i' - (q_i - p_i')p = q_i \tag{2}$$

for  $p_i'$ . For  $i = m$  the equation becomes

$$p + q_m - p_m' - (q_m - p_m')p = q_m, \tag{3}$$

And  $p_m' = \frac{p(1-q_m)}{1-p}$  is the solution of this equation. Since by Step 4 we have  $\Pr(e_m \in G_{n,p}^m) = p$  and  $\Pr(e_m \in G_2^m) = q_m - p_m'$ , therefore, we get that the left-hand side of (3) is precisely the probability of the event  $\{e_m \in G_{n,p}^m \cup G_2^m\}$ . By (3), this probability is equal to  $q_m$ , which is set to  $q_m = \Pr(h_m = 1) = \Pr(e_m \in G_{n,p}^m)$  in Step 2. This means that  $G_{n,p}^m \cup G_2^m \sim \tilde{G}_{n,p}^m$  as desired. For the induction step, assume that the claim is true for

$i + 1$ , i.e.,  $G_{n,p}^{i+1} \cup G_2^{i+1} \sim \tilde{G}_{n,p}^{i+1}$  holds. In Step 4, edge  $e_i$  is added to  $G_{n,p}^{i+1}$  with probability  $p$ . It is also added to  $G_2^{i+1}$  with probability  $q_i - p_i'$ . Therefore, just like in the base case, we get that  $p + q_i - p_i' - (q_i - p_i')p = \Pr(e_i \in G_{n,p}^{i+1} \cup G_2^{i+1})$ . We already know that  $p_i'$  satisfies the equation (2), so  $e_i$  is added to  $G_{n,p}^{i+1}$  with probability  $q_i = \Pr(h_i = 1 | h_{i+1}^m = a_{i+1}^m)$ , given the already generated part, represented by  $a_{i+1}^m$ . By the inductive assumption,  $h_{i+1}^m$  is distributed as  $\tilde{G}_{n,p}^{i+1}$ , which is the truncated version of  $\tilde{G}_{n,p}$ , keeping only the  $\geq i+1$  indexed edges. Hence, for  $h_{i+1}^m$ , we can write by the chain rule of conditional probabilities:

$$\Pr(h_{i+1}^m = a_{i+1}^m) = \Pr(h_m = a_m) \prod_{j=i+1}^{m-1} \Pr(h_j = a_j | h_{j+1}^m = a_{j+1}^m).$$

After processing  $e_i$  (i.e., adding it with probability  $q_i$ ), we get

$$\begin{aligned} \Pr(h_i^m = a_i^m) &= \Pr(h_i = a_i | h_{i+1}^m = a_{i+1}^m) \Pr(h_{i+1}^m = a_{i+1}^m) \\ &= \Pr(h_i = a_i | h_{i+1}^m = a_{i+1}^m) \Pr(h_m = a_m) \prod_{j=i+1}^{m-1} \Pr(h_j = a_j | h_{j+1}^m = a_{j+1}^m) \\ &= \Pr(h_m = a_m) \prod_{j=i}^{m-1} \Pr(h_j = a_j | h_{j+1}^m = a_{j+1}^m), \end{aligned}$$

which, by the chain rule, is indeed the distribution of  $\tilde{G}_{n,p}^i$ , completing the induction.

Thus, at the end, a realization  $a = a_i \in \{0, 1\}^m$  of  $\tilde{G}_{n,p}$  is generated with probability  $\Pr(h_1^m = a) = \Pr(h_m = a_m) \prod_{j=1}^{m-1} \Pr(h_j = a_j | h_{j+1}^m = a_{j+1}^m)$ , indeed creating  $\tilde{G}_{n,p}$  with its correct probability. Therefore, we get  $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$ , so  $\tilde{G}_{n,p}$  arises by adding edges to  $G_{n,p}$ , which cannot destroy a monotone property. This implies the statement of the Theorem. As a sample application, we can claim that the random graph described in example 3, asymptotically has a maximum clique of size at least  $2 \log_{10/3} n$ , chromatic number at least  $n / \log_b n$  with  $b = 1/(1-p) = 10/7$ , minimum dominating set at most  $\log_b n$ , also with  $b = 10/7$ , and diameter at most  $\frac{\log n}{\log(3n/10)}$ . These follow from the known results about the Erdős-Rényi random graph  $G_{n,p}$ , complemented with our Theorem. Note that such bounds would be very hard to prove directly from the definition of the edge dependent random graph model.

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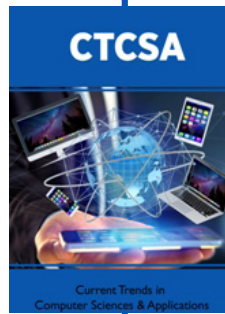


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