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Review Article

Comparison Between Second Zagreb Eccentricity Index and Eccentric Connectivity Index of Graphs

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Abstract

For a graph G, the second Zagreb eccentricity index $E_2(G)$ and eccentric connectivity index $\xi^c(G)$ are two eccentricity-based invariants of graph G. In this paper we prove some results on the comparison between $\frac{E_2(G)}{m}$ and $\frac{\xi^c(G)}{n}$ of connected graphs G of order n and with m edges.

Keywords: Graph; Eccentricity (of vertex); Second Zagreb eccentricity index; Eccentric connectivity index

Introduction

Throughout this paper we only consider the finite, undirected, simple and connected graphs. The degree of v ∈ V(G), denoted by $\deg_G(v)$, is the number of vertices in G adjacent to v. For any two vertices u; v in a graph G, the distance between them, denoted by $d_G(u,v)$ is the length of a shortest path connecting them in G. As usual, let Sn, Pn, Cn, Kn be the star graph, path graph, cycle graph and complete graph, respectively, on n vertices. Other undefined notations and terminology on the graph theory can be found in [1]. For any vertex of graph G, the eccentricity $\varepsilon_G(v)$ (or $\varepsilon(v)$ for short) is the maximum distance from v to other vertices of G, i.e., $\varepsilon_G(v) = \max_{u \neq v} d_G(u,v)$. The eccentricity of a vertex is an important parameter in pure graph theory. The radius of a graph G is denoted by r(G) and defined by r = r(G) = {min $\varepsilon(v)$ | v ∈ V(G)}. Also, the diameter of G, denoted by d(G), is the maximum distance between vertices of a graph G and hence d = d(G) = max{ $\varepsilon(v)$ | v ∈ V(G)}. A vertex v with $\varepsilon_G(v)$ = r(G) is called a central vertex in G. A graph G with d(G) = r(G) is called a self-centered graph. A graph which contains only two non-central vertices is called almost self-centered graph [2] (ASC graph for short). Moreover, the eccentricity is also applied in chemical graph theory. There are several eccentricity-based topological indices, including the second Zagreb eccentricity index $E_2(G)$ [3] and eccentric connectivity index $\xi^c(G)$ [4], of graphs G where $E_2(G) = \sum_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v)$ and $\xi^c(G) = \sum_{v\in V(G)} \deg_G(v)\varepsilon(v)$.

In particular, we have $\xi^c(G) = \sum_{uv\in E(G)} (\varepsilon(u)+\varepsilon(v))$ or any graph G. In this paper we prove some comparison results between $\frac{E_2(G)}{m}$ and $\frac{\xi^c(G)}{n}$ of connected graphs G of order n with m edges. Main results

In this we prove several results on the comparison between $\frac{E_2(G)}{m}$ and $\frac{\xi^c(G)}{n}$ of graphs G. Firstly we present two useful lemmas.

Lemma 2.1. ([5]) Let G be a connected graph of order n with maximum degree Δ. If $\Delta = n-1$ then $E_2(G) = \xi^c(G)$ .Otherwise, $E_2(G) \geq \xi^c(G)$ with equality holds if and only if G is a 2-SC graph.

Lemma 2.2. ([6]) If u and v are two adjacent vertices of a connected graph G, then $|\varepsilon(u)-\varepsilon(v)\leq 1|$.

Denote by $\zeta_n(m;d)$ the set of connected graphs of order n with m edges and diameter d.

Theorem 2.3. Let $G\in\zeta_n(m,d)$ with n > 5 and $d\leq 2$. Then $\frac{E_2(G)}{m} < \frac{\xi^c(G)}{n}$. Proof. If d = 1, $G\in\zeta_n(m,d)$ contains a single graph Kn with $m=\binom{n}{2}$ and $E_2(K_n)=\binom{n}{2} < n(n-1) = \xi^c(K_n)$. Then our result follows. Next it suffices to consider the case when d = 2. If G has maximum degree $\Delta = n-1$ by Lemma 2.1, we have $E_2(G) < \xi^c(G)$ for any graph $G\in\zeta_n(m,d)$ .Moreover, we have $m \geq n-1$ If m = n-1, then $G\cong S_n$ with $\frac{E_2(G)}{m} = 2 < \frac{\xi^c(G)}{n} = \frac{3(n-1)}{n}$ for any n ≥ 5.

Moreover, $\frac{E_2(G)}{m} < \frac{\xi^c(G)}{n}$ holds clearly form ≥ n. If $\Delta \le n-2$ then G is a 2-SC graph. By Lemma 2.2, G is never a tree. Therefore m ≥ n with equality holding if and only if $G \cong C_4$ or $G \cong C_5$. Consider that n > 5, m > n holds immediately. It follows that $\frac{E_2(G)}{m} = 4 < \frac{4m}{n} = \frac{2\sum_{uv\in V(G)} \deg_G(v)}{n} = \frac{\xi^c(G)}{n}$. This completes the proof of the theorem.

In the following we consider the graphs $G \in \zeta_n(m,d)$ with diameter d ≥ 3.

Theorem 2.4. Let $G \in \zeta_n(m,d)$. with d ≥ 3, n > 5 be a tree or a unicyclic graph. Then $\frac{E_2(G)}{m} > \frac{\xi^c(G)}{n}$

Proof. If d ≥ 3, then $\Delta(G) \le n-2$. From Lemma 2.1. we have $E_2(G) > \xi^c(G)$. Note that m ≤ n for any tree or unicyclic graph G. Thus, it follows $\frac{E_2(G)}{m} \ge \frac{E_2(G)}{n} > \frac{\xi^c(G)}{n}$. finishing the proof of the theorem. Next, we consider the case m > n. In the following theorem we give a sufficient condition for the graph G of order n with $\frac{E_2(G)}{m} \ge \frac{\xi^c(G)}{n}$.

Theorem 2.5. Let $G \in \zeta_n(m,d)$ with d ≥ 3, m = n + t and $1 \le t \le \frac{n}{2}$. If r(G) ≥ 3, then $\frac{E_2(G)}{m} \ge \frac{\xi^c(G)}{n}$, with equality holding if and only if G is a self-centered graph with r(G) = 3.

Proof. Making a difference, we have

$$\frac{E_2(G)}{m} - \frac{\xi^c(G)}{n} = \frac{nE_2(G)-(n+t)\xi^c(G)}{n(n+t)}.$$

Set $\Delta_1 = nE_2(G) - (n+t)\xi^c(G)$. From Lemma 2.2, we have

$$\Delta_1 = \sum_{uv\in E(G)} [n\varepsilon(u)\varepsilon(v) - (n+t)(\varepsilon(u)+\varepsilon(v))]$$

$$\Delta_1 = \sum_{uv\in E(G),\varepsilon(v)=\varepsilon(u)} \left[n\varepsilon^2(u) - 2\varepsilon(u)(n+t)\right] + \sum_{uv\in E(G),\varepsilon(v)=\varepsilon(u)+1} \left[n\varepsilon(u)(\varepsilon(u)+1) - (n+t)(2\varepsilon(u)+1)\right]$$

Since r(G) ≥3 and $1 \le t \le \frac{n}{2}$, we have

$$n\varepsilon^2(u) - 2\varepsilon(u)(n+t) \ge 9n - 6(n+t) = 3(n-2t) \ge 0, n\varepsilon(u)(\varepsilon(u)+1) - (n+t)(2\varepsilon(u)+1)$$
$$= \varepsilon(u)[n\varepsilon(u) - n - 2t] - n - t \ge 3(2n-2t) - n - t = 5n - 7t > 0$$

Therefore, $\Delta_1 \ge 0$ with equality holding if and only if $\varepsilon(u) = 3$ for each vertex $u \in V(G)$ that is, G is a self-centered graph with radius 3. This completes the proof of the theorem.

For, $G \in \zeta_n(m,d)$ with d ≥ 3, r = 2 and $1 \le t \le \frac{n}{2}$ considering that $r(G) \le d(G) \le 2r(G)$ we have

d(G) = 3 or d(G) = 4. In this case, the value of Δ_1 may be negative, zero or positive. Let

$\varepsilon_1 = \{uv : uv \in E(G), \varepsilon(u) = \varepsilon(v) = 2\}, \varepsilon_2 = \{uv : uv \in E(G), \varepsilon(u) = 2, \varepsilon(v) = 3\}$, $\varepsilon_3 = \{uv : uv \in E(G), \varepsilon(u) = \varepsilon(v) = 3\}, \varepsilon_4 = \{uv : uv \in E(G), \varepsilon(u) = 3, \varepsilon(v) = 4\}$, $\varepsilon_5 = \{uv : uv \in E(G), \varepsilon(u) = \varepsilon(v) = 4\}$.

Denote by m_i the cardinality of ε_i and $i \in \{1,2,3,4,5\}$. Then

$$\Delta_1 = \sum_{uv\in E(G)} [n\varepsilon(u)\varepsilon(v) - (n+t)(\varepsilon(u)+\varepsilon(v))]$$

$$\Delta_1 = -4tm_1 + (n-5t)m_2 + (3n-6t)m_3 + (5n-7t)m_4 + (8n-8t)m_5$$

In the following result we present some comparison results for ASC graphs.

Theorem 2.6. Let $G \in \zeta_n(m,d)$ with d = 3, r = 2, m = n + t, t ≥ 1 where $n < \frac{5t + 2\sqrt{41t^2+12t+4}}{2}$. If G is an ASC graph, then $\frac{E_2(G)}{m} < \frac{\xi^c(G)}{n}$

Proof. If G is an ASC graph with d = 3, r = 2, from the structure of ASC graph, we have $m_2 \le n-2, m_3 = 0$, that is, $m_1 \ge t+2$. If $n \le 5t$, clearly, we have $\Delta_1 \le 0$. For n > 5t, we have

$$\Delta_1 \le -4t(t+2) + (n-5t)(n-2)$$
$$= n^2 - (5t+2)n - 2t(2t-1)$$
$$< 0$$

holds if and only if $\frac{5t+2-\sqrt{41t^2+12t+4}}{2} < n < \frac{5t+2+\sqrt{41t^2+12t+4}}{2}$

Note that

$\frac{5t+2-\sqrt{41t^2+12t+4}}{2} < 0$ Thus $\Delta_1 < 0$ is equivalent that $n < \frac{5t+2+\sqrt{41t^2+12t+4}}{2}$ with $t \ge 1$. Therefore the result holds immediately.

It is much interesting to search for more generalized graphs G with different comparison results between $\frac{E_2(G)}{m}$ and $\frac{\xi^c(G)}{n}$ which can be a topic for further research in the future.

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