Current Trends on Biostatistics & Biometrics
DOI: 10.32474/CTBB.2020.02.000137
ISSN: 2644-1381
Research Article

On Domination Number of Mixed-Grid Graph

Ramy Shaheen*

Department of Mathematics Faculty of Science, Tishreen University, Syria

***Corresponding author:** Ramy Shaheen, Department of Mathematics Faculty of Science, Tishreen University, Lattakia, Syria

Received: February 03, 2020 **Published:** February 27, 2020

Abstract

A mixed graph GM (V, E, A) is a graph containing unoriented edges (set E) as well as oriented edges (set A), referred to as arcs. In this paper we calculate the domination number of the Cartesian product of a path Pm with directed path Pn (mixed-grid graph $p_m \times \vec{p}_n$) for some values of m and arbitrary n.

Keywords: graph, directed graph, Cartesian product, path, directed path, mixed graph, mixed-grid graph, dominating set, domination number.

Introduction

All graphs and digraphs are assumed to be loop less and without duplicate edges or arcs. A mixed graph GM (V,E,A) is a graph containing unoriented edges (set E) as well as oriented edges (set A), referred to as arcs. This notion was first introduced in [1]. Let $G=(V_1,E)$ be a graph and $D=(V_2,A)$ be a digraph. The Cartesian product $G\times D$ is the mixed-graph with vertex set $V(G\times D)=V_1(G)\times V_2(D)$ and edge (arc) set is $((u_1,v_1),(u_2,v_2)\in E(G\times D)$ if and only if either $v_1=v_2$ and $(u_1,u_2)\in E(G)$ or $u_1=u_2$ and $v_1,v_2)\in A(D$. A subset S of the vertex set $V(G\times D)$ is a dominating set of $(G\times D)$ if for each vertex $V\in G\times D$ there exists a vertex $u\in s$ such that (u, v) is an edge (arc) of $G\times D$. The domination number of $G\times D$, $\gamma G\times D$, is the cardinality of the smallest dominating set of $G\times D$. Let Pm be a path with vertex set $V(P_m)=\{1,2,\dots,n\}$, and edge set $E(P_m)=\{(i,i+1):1\le i\le n-1\}$ and let $\vec{P}_n$ be a directed path with vertex set $V(\vec{P}_n)=\{1,2,\dots,n\}$, and arc set $A(P_n)=\{(i,i+1):1\le i\le n-1\}$. Then for Cartesian product P_m and $\vec{P}_n$ is mixed-grid graph $P_m\times\vec{P}_n$ with $V(P_m\times\vec{P}_n)=\{(i,j):1\le i\le m,1\le j\le n\}$, such that there is an arc from (i,j) to (p,q) if and only $fi=p$ and $q-j=1$ and is an edge from (i,j) to (p,q) if and only if $j=q$ and $p-i=1$. The i'th row of $V(P_m\times\vec{P}_n)$ is $R_i=\{\text{is a directed path}\vec{P}_n(i,j): j=1,2,\dots,n\}$ The j'th column $K_j=\{(i,j)\text{is a path}P_m(i,j): i=1,2,\dots,m\}$. If S is a dominating set for $P_m\times\vec{P}_n$, then we denote $W_j=S\cap K_j$.

Let $S_j=|W_j|$, where the sequence $(s_1,s_2,\dots,s_n)$ is called a dominating sequence corresponding to S. For $1\le i\le n$, the vertices of j-column are dominated by vertices of j-column or (j-1) column. The vertices of the first column are dominated only by [2] vertices of K1. Also, for 1 ≤ i ≤m, the vertices of i-row are dominated by vertices of i-row or $(i-1)-$ row. The vertices of the first row are dominated only by vertices of R1 or R2. Thus, the following is true $\gamma(Pm\times Pn)\le\gamma(Pm\times Pn)\le\gamma(Pn\times Pn)$. For finding domination number of grid graphs $P_m\times\vec{P}_n$, Jacobson and Kinch in [2], were calculated the domination number of cartesian product of undirected paths P_m and P_n for $m=1,2,3,4$. The cases $m=5,6$ were calculated by Chang and Clark [3]. Also, Chang et al. [4], established the upper bounds of cartesian product of undirected paths Pm and Pn for $5\le m\le 10$ and arbitrary n. In [5], Gravier and Mollard given un upper and lower bounds of general cartesian product of two undirected paths. Goncalves et al. [6] proved Chang's conjecture saying that for every $16\le n\le m,\gamma(P_m\times\vec{P}_n)=[(n+1)(m+2)/5]-4$. For domination number of directed grid graphs, Liu et al. [7], they studied the domination number of $\vec{P}_m\times\vec{P}_n$ for m = 2, 3, 4, 5, 6 and arbitrary n. Also, in [8] the author studied the domination number of $\vec{P}_m\times\vec{P}_n$ for arbitraries m and n.

Main results

In this section we calculate the domination number of the Cartesian product of a path P_m and a directed path $\vec{P}_n$ for some values of m and arbitrary n.

Observation

Since for each vertex $(i,j) \in V(P_m \times \vec{P}_n)$ has two undirected degrees in $V(K_j)$ one outdegree in $V(K_{j+1})$ and one indegree from $V(K_{j-1})$, then can it dominates at must four vertices of $P_m \times \vec{P}_n$ with itself. Thus, implies that $\gamma(P_m \times \vec{P}_n) \geq mn/4$.

Observation

Let S be a dominating set of $P_m \times \vec{P}_n$. Since the vertices of the first column are dominated only by vertices of K_1. Also, for $1 \leq i \leq n$, the vertices of j-column are dominated by vertices of j-column or (j-1)-column. Then the following are holds:

1. $s_1 \geq [m/3]$.
2. $s_j + 3s_{j+1} \geq m$ for all j=1,....,n .
3. $s_{j-1} + 4s_j + s_{j+1} \geq 2m$ for all j = 2,....,n

Lemma

There is a minimum dominating set S for $P_m \times \vec{P}_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$ such that, for all $j = 1, 2, \ldots, n$, , is $[(m+2)/4] \leq s_j \leq [m/2]$.

Proof: Let S be a minimum dominating set for $P_m \times \vec{P}_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$. Assume that for some j, s_j is large. Then we modify S by moving some vertices from column j to column $j+1$, such that the resulting set is still dominating set for $P_m \times \vec{P}_n$ (because each vertex in $S \cap K_j$ is dominates only vertices from K_j and K_{j+1}). For $1 \leq i \leq m$ and $1 \leq i \leq n$, let $W = S \cap \{(i,j), (i+1,j), (i+2,j), (i+3,j)\}$. If $W \geq |3|$, then we have three cases.

Case: If $\{(i,j),(2,j)\} \subseteq S$ or $\{(m-1, j(m,j)\} \subseteq S$. Then we can moving $(1,j)$ to $(1, j+1)$ or (m,j) to $(m, j+1)$. Furthermore, S is still dominating set of $P_m \times \vec{P}_n$.

Case: $|W| = 4$, then we put

$$S_1 = (S - W) \cup \{(i,j), (i+1, j+1), (i+2, j+1).(i+3, j)\}$$

Case: $|W| = 3$ then we have two sub cases

a. **SubCase**: $|W| = 3$ and $W = S \cap \{(i,j),(i+1,j),(i+2,j),(i+3,j)\} = \{(i,j),(i+1,j),(i+2,j)\}$ or W = {(i+1, j), (i+2, j), (i+3, j)}. Two cases are similar by symmetry, for the first we put $S_1 = (S-W) \cup \{(i,j),(i+2,j),(i+1,j+1)\}$ and for the second we put $S_1 = (S-W) \cup \{(i,j),(i+3,j),(i+2,j+1)\}$

b. **Subcase**: $|W| = 3$ and $W = S \cap \{(i,j),(i+1,j),(i+2,j),(i+3,j)\} = \{(i,j),(i+1,j),(i+3,j)\}$ or $W = \{(i,j),(i+2,j),(i+3,j)\}$. Also, two cases are similar by symmetry. Then we change S, respectively as follows: $S_1 = (S-W) \cup \{(i,j),(i+3,j)(i+1,j+1)\}$, $S_1 = (S-W) \cup \{(i,j),(i+3,j),(i+2,j+1)\}$, see (Figure 1) for cases 2, 3. We repeat this process if necessary eventually leads to a dominating set with required properties. Also, we get S1 is a dominating set for $P_m \times \vec{P}_n$ with $|S| = |S_1|$. Thus, we can assume that every four consecutive vertices of the j'th column include at most two vertices of S. This implies that $S_j \leq [m/2]$, for all $1 \leq j \leq n$.To prove the lower bound, we suppose that $|K_j \cap D|$ is be a maximum, i.e $S_j = [m/2]$ Then there at must $m - [m/2]$ vertices in K_{j+1} are not dominated (Figure 2). By maximality of S_j, does not have three successive vertices do not belong to S (i.e., does not exist three successive vertices K_{j+1} from does not dominated by vertices from $S \cap K_j$). Thus, implies that $S_{j+1} \geq [(m - [m/2])/2] = [(m+2)/4]$ By

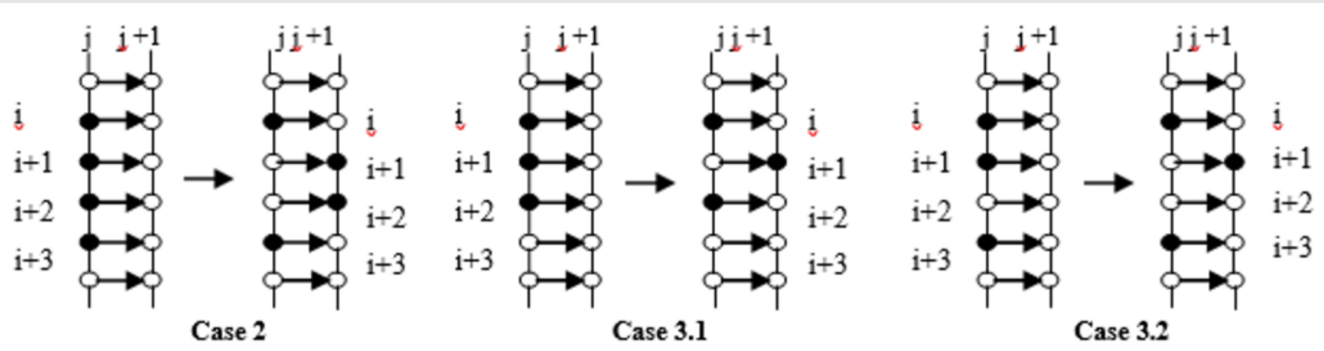

Figure 1: Modify S.

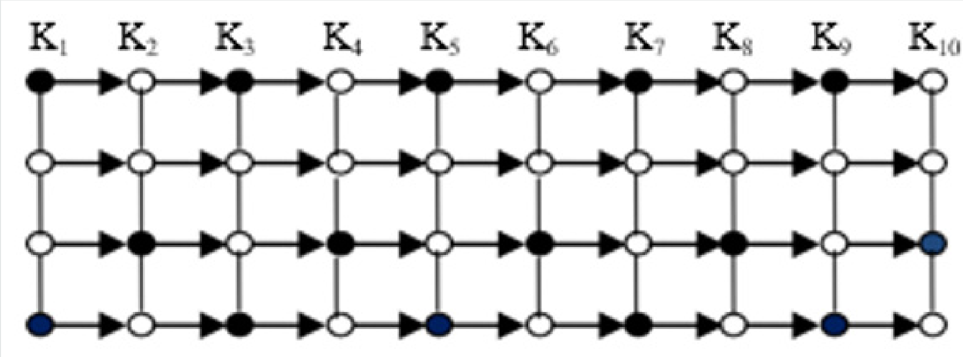

Figure 2: A dominating function of $P_4 \times P_{10}$.

c. **Lemma:** always we have a minimum dominating set S with dominating sequence $(s_1, s_2, ..., s_n)$, such that $[(m+2)/4] \le S_j \le [m/2]$, for all $j = 1, 2, ..., n$

Proposition: Clearly, $\gamma(P_1 \times \vec{P}_n) = \gamma(\vec{P}_n) = \left[\frac{n}{2}\right]$

Theorem $\gamma(P_2 \times \vec{P}_n) = n$

Proof. Let S be a set defined as follows $S = \left\{(1, 2j-1) : 1 \le j \le \left[\frac{n}{2}\right]\right\} \cup \left\{(2, 2j) : 1 \le j \le \left[\frac{n}{2}\right]\right\}$ We have $|S| = n$, also S is a dominating set of $P_2 \times \vec{P}_n$. By Observation 2, we have $S_j + 3S_{j+1} \ge m$. Which implies that, if $S_j = 0$ is $S_{j+1} = 2$ and if $S_j = 1$ is $S_{j+1} \ge 1$. Then we get $\gamma(P_2 \times \vec{P}_n) = \sum_{j=1}^{n} S_j \ge n$

Thus $\gamma(P_2 \times \vec{P}_n) = n$

Theorem: $\gamma(P_3 \times \vec{P}_n) = n$

Proof: We define S as follows: $S = \{(2, j) : 1 \le j \le n\}$

Certainly, S is a dominating set of $P_3 \times \vec{P}_n$ with $|S| = n$

Also, we have $\gamma(P_3 \times \vec{P}_n) \ge \gamma(P_2 \times \vec{P}_n) = n$ Thus

$$\gamma(P_4 \times \vec{P}_n) = n$$

Proposition: $\gamma(P_4 \times \vec{P}_n) \le \left[\frac{3n}{2}\right]$

Proof. Here, we define S as follows:

$$S = \{(1, 4j-3), (4, 4j-3) : 1 \le j \le [n/2]\} \cup \{(3, 4j-2) : 1 \le j \le \left[\frac{n}{2}\right]\}$$

. Certainly S is a dominating set $P_4 \times \vec{P}_n$ (see Figure 3 for $\gamma(P_4 \times \vec{P}_{10})$), . Also, we have

$$|S| = 2[n/4] + 2[(n-1)/4] + [(n-2)/4] + [(n-3)/4] = [3n/2]$$

$$\gamma(P_4 \times \vec{P}_n) \le \left[\frac{3n}{2}\right].$$

Proposition:

a. There only two cases of $(S_j, S_{j+1}) = (1,1)$ such that $K_j \cap S = \{(1, j)\}$ and $K_{j+1} \cap S = \{(3, j+1)\}$ or $K_j \cap S = \{(4, j)\}$ and $K_{j+1} \cap S = \{(2, j+1)\}$ Furthermore, two cases are similar by symmetry. We consider the first case.

b. The case $(S_j, S_{j+1}, S_{j+2}) = (1,1,1)$ is not possible.

c. The case $(S_1, S_2, S_3) = (2,1,1)$ is not possible.

d. If $K_2 \cap S = \{(1,2)\}$ then $S_1 = 3$

e. If $K_3 \cap S = \{(1,3)\}$ then $S_1 + S_2 = 4$.

Proof: Immediately by drawing.

Proposition: If $(S_j, S_{j+1}) = (1,1)$, then $(S_{j-2}, S_{j-3}) = (2,2)$ or $(S_{j-2r}, S_{j-2r+1}) = (1,2)$ {where $1 \le r$ and $1 \le j - 2r$. Furthermore, the second case repeated until exist two columns before K_j, K_{j+1} ,contains four vertices from S.

Proof: From Proposition 4.5.1 (1), we have $K_j \cap S = \{(1, j)\}$ and $K_{j+1} \cap S = \{(3, j+1)\}$. This implies that $K_{j-1} \cap S = \{(3, j-1), (4, j-1)\}$ and then $\{(1, j-2)\} \subseteq K_{j-2} \cap S$. If $|K_{j-2} \cap S| = 2$ then $S_{j-1} + S_{j-2} = 4$ otherwise $|K_{j-2} \cap S| = 1$ and $K_{j-3} \cap S = \{(3, j-3), (4, j-3)\}$, $K_{j-4} \cap S = \{(1, j-4)\}$ and repetition this case until $(S_1, S_2) = (1,2)$ or $(S_2, S_3) = (1,2)$ {where j is odd or even, respectively}.

Theorem: $\gamma(P_4 \times \vec{P}_n) = \left\lceil \frac{3n}{2} \right\rceil$

Proof. By Lemma 2.1, $1 \le S_j \le 2$ for all $j = 1,\dots,n$. We have two cases:

Case 1. $S_j + S_{j+1} \ge 3$ for j = 1, ..., n-1. Then we get $\gamma(P_4 \times \vec{P}_n) = \sum_{j=1}^{n} S_j \ge \left\lceil \frac{3n}{2} \right\rceil$ {because $S_1 \ge 2$ }.

Case 2. $S_j + S_{j+1} = 2$ for some $1 \le j \le n-1$. Thus, we have $(S_j + S_{j+1}) = (1,1)$. By Proposition 2.3(1), we consider the first case $K_j \cap S = \{(1, j)\}$ and $K_{j+1} \cap S = \{(3, j+1)\}$. There are two subcases

Sub Case $S_{j-2r} + S_{j-2r+1} = 4$ {where $4 \le j \le -2r \le j-2$ }. Then we can assume that each two columns including at least three vertices from S and then $\gamma(P_4 \times \vec{P}_n) = \sum_{j=1}^{n} S_j \ge \left\lceil \frac{3n}{2} \right\rceil$

Sub Case $(S_{j-2}, S_{j-1}) = (S_{j-4}, S_{j-3}) = \dots = (S_{j-2r} + S_{j-2r+1}) = (1,2)$. Then we study the cases:

a. If $j - 2r = 1$ i.e., $S_1 = 1$ and this impossible.

b. If $j - 2r = 2$. Then $K_2 \cap S = \{(1,3)\}$ and $\{(3,1),(4,2)\} \subseteq K_2 \cap S$. But, needs $(1,1) \in S$ or $(2,1) \in S$ for dominate (1, 1). Furthermore $S_1 + S_2 = 4$.

c. If $j - 2r = 3$. Then $K_3 \cap S = \{(1,3)\}$ and $\{(3,2),(4,2)\} \subseteq K_2 \cap S$. Then needs $(1,1) \in S$ and $(3,1) \in S$ or $(4,1) \in S$, also $S_1 + S_2 = 4$.

Now, we conclude that for all cases, when exist the case $(S_j, S_{j+1}) = (1,1)$ there is the case $S_{j-2r} + S_{j-2r+1} \ge 4$ before repeated the case $(S_q, S_{q+1}) = (1,1)$. Thus, we can assume that each four columns including at least 6 vertices from S. Finally, get $\gamma(P_4 \times \vec{P}_n) = \sum_{j=1}^{n} S_j \ge \left\lceil \frac{6n}{4} \right\rceil = \left\lceil \frac{3n}{2} \right\rceil$ Finally, Proposition 2.2 together with the last result gets

$$\gamma(P_4 \times \vec{P}_n) = \left\lceil \frac{3n}{2} \right\rceil$$

Theorem: $\gamma(P_5 \times \vec{P}_n) = \left\lceil \frac{3n}{2} \right\rceil + 1$

Proof: Let S defined as follows:

$$S\{(1,1),(3,1),(5,1)\} \cup \{(3,2j) : 1 \le j \le \left\lceil n/2 \right\rceil\} \cup \{(1,2j+1),(5,2j+1) : 1 \le j \le [(n-1)/2]\}$$

We can check that S is a dominating set of $P_5 \times \vec{P}_n$ (see Figure 3 for $\gamma(P_5 \times \vec{P}_{12})$), with

$S = 3 + [n/2] + 2[(n-1)/2] = [3n/2] + 1$ We need to the following fact:

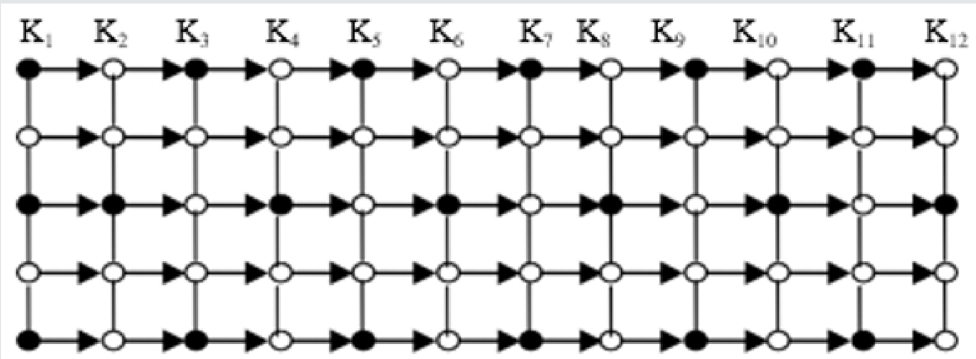

Figure 3: A dominating function of $P_5 \times P_{12}$.

Fact 1.

1. There is one possible for $(S_j, S_{j+1}, S_{j+2}) = (1,2,1)$. Furthermore, if $(S_j, S_{j+1}, S_{j+2}) = (1,2,1)$ then $K_j \cap S = \{(3, j)\}$, $K_{j+1} \cap S = \{(1, j+1),(5, j+1)\}$ and $K_j \cap S = \{(3, j+2)\}$.

2. $(S_1, S_2, S_3) = (2,2,1)$ in not possible.

3. $(S_1, S_2, S_3) = (2,1,2)$ in not possible. Furthermore, $S_1 + S_2 \ge 4$ and $S_1 + S_2 + S_3 \ge 6$

Proof. 1). By Observation 2.2, we have $(S_j, S_{j+1}) = (1,1)$ is not possible. But, $(S_j, S_{j+1}) = (1,2)$ or (2, 1) are possible, also $(S_j, S_{j+1}, S_{j+2}) = (1,2,1)$ is possible. Thus for $S_{j+2} = 1$ we must have $K_{j+1} \cap S = \{(1, j+1),(2, j+1)\}$, $\{(4, j+1),(5, j+1)\}$ or $\{(1, j+1),(5, j+1)\}$. For the first and second needs $S_j = 2$, we rejected. Then for the case $K_{j+1} \cap S = \{(1, j+1),(5, j+1)\}$, we get $S_j = 1$ and $S_{j+2} = 1$, furthermore $K_j \cap S = \{(3, j)\}$ and $K_j \cap S = \{(3, j+2)\}$.

2). From (1), if $S_3 = 1$ then we have $K_2 \cap S = \{(1,2),(2,2)\},\{(4,2),(5,2)\}$ or $\{(1,2),(5,2)\}$. For all these cases $S_1 = 2$ is not possible.

3) Immediately from prove (1).

Now, From Lemma 2.1, we have $1 \le S_j \le 3$. Also, Observation 2.2, gets (3, 1, 1) is not possible. By Fact 1, we include that $S_1 + S_2 + S_3 \ge 6$, furthermore if $S_j = 2$ then $\sum_{j}^{j-r} s_j \ge \left[\frac{3(r+1)}{2}\right]$ for r ≥ 2. Thus

If n=0(mod2) then $\gamma(P_5 \times \vec{P}_n) = \sum_{j=1}^{n} S_j = S_1 + S_2 + \sum_{j=3}^{n} S_j \ge 4 + \frac{3(n-2)}{2} = \left[\frac{3n}{2}\right] + 1$

Forn=1(mod2) $\gamma(P_5 \times \vec{P}_n) = \sum_{j=1}^{3} S_j = \sum_{j=4}^{n} S_j \ge 6 + \frac{3(n-3)}{2} = \left[\frac{3n}{2}\right] + 1$

By recent results with (1), we get the required.

Theorem $\gamma(P_6 \times \vec{P}_n) = n$.

Proof. Let S defined as follows: S = $S = \{(2,1),(5,j) : 1 \le j \le n\}$. Definitely S is a dominating set of $P_6 \times \vec{P}_n$ with $\gamma(P_6 \times \vec{P}_n) \le |S| = 2n$.

By Observation 2.2, $S_1 \ge 2$ and $(S_j, S_{j+1}) = (1,2)$ is not possible. This implies that $\gamma(P_6 \times \vec{P}_n) = \sum_{j=1}^{n} S_j \ge 2n$ Thus we get $\gamma(P_6 \times \vec{P}_n) = \sum_{j=1}^{n} S_j = 2n$

Theorem $\gamma\left(P_7 \times \vec{P}_n\right) = 2n + 2$

Proof. For $P_7 \times \vec{P}_n$ we define S as follows:

$$S = \{(1,1),(3,1),(5,1),(7,1),\} \cup \{(3,2j),(7,2j) : 1 \le j \le \lfloor n/2 \rfloor\} \cup \{(1,2j+1),(5,2j+1) : 1 \le j \le \lfloor (n-1)/2 \rfloor\}$$

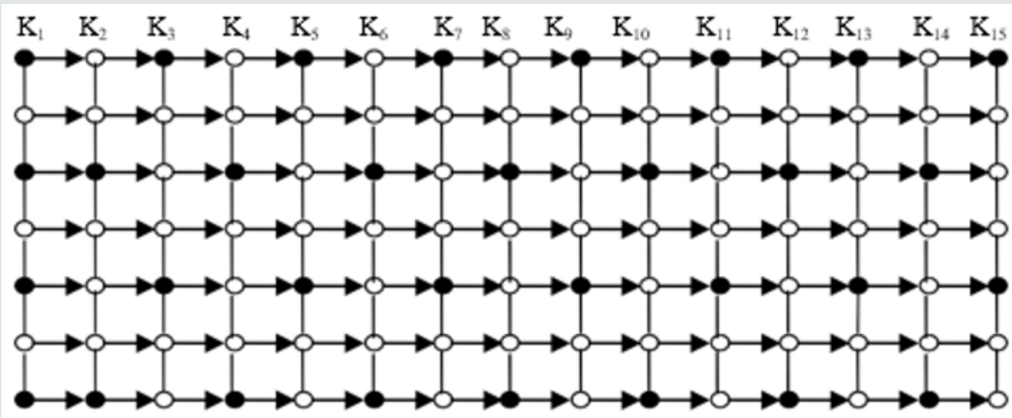

Figure 4: A dominating function of $P_7 \times P_{15}$.

We have S is a dominating set of $P_7 \times \vec{P}_n$ (see (Figure 4), for $\gamma\left(P_7 \times \vec{P}_{15}\right)$),with $|s| = 4 + 2\lfloor n/2 \rfloor + 2\lfloor (n-1)/2 \rfloor = 2n + 2$

We need to the following fact.

a. **Fact 2.**

b. $s_1 \ge 3$

c. $s_1 \ge 3$

d. $(s_1, s_2) = (4,1)$ and $(s_1, s_2, s_3) = (3,2,2)$

e. $s_1 + s_2 \ge 6$ or $s_1 + s_2 + s_3 \ge 6$

$\sum_{d=j}^{j+k} s_d \ge 2(k+1)$, where k ≥1.

Proof. 1). It is clear from Observation 2.2.

2) Immediately by drawing.

3) From (2), it's clear.

4) By Observation 2.2, we have $s_j + 3s_{j+1} \ge m$. This implies that $(s_j, s_{j+1}) = (3,1)$ is not possible. Then we get the required for (4).

Now, by Fact 2, We conclude that

$$\gamma\left(P_7 \times \vec{P}_n\right) = \sum_{j=1}^{n} s_j \ge s_1 + s_2 + \sum_{j=3}^{n} s_j \ge 6 + 2(n-2) = 2n + 2 \text{ (or)}$$

$$\gamma\left(P_7 \times \vec{P}_n\right) = \sum_{j=1}^{n} s_j \ge s_1 + s_2 + s_3 + \sum_{j=4}^{n} s_j \ge 8 + 2(n-3) = 2n + 2$$

Thus, by (2) together with the last result gets $\gamma\left(P_7 \times \vec{P}_n\right) = 2n + 2$.

References

1. Sotskov YN, Tanaev VS (1976) Chromatic polynomial of a mixed graph. Vestsi Akademie Navuk BSSR, Seriya Fiz. Mat Navuk Russian pp. 620-623.

2. Chang T, Clark W (1993) The domination number of the 5×n and 6×n grid graphs. J of Graph Theory 17(1): 81-107.
3. Chang T, Clark W, Hare E (1994) Domination number of complete grid graphs. I Ars Combinatoria 38: 97-111.
4. Gravier S, Modlard M (1997) On domination number of Cartesian product of paths complete. Discrete applied mathematics 80: 2497-2550.
5. Goncalves D, Pinlou A, Rao M, Thomasse S (2011) The Domination Number of Grids. SIAM J Discrete Math 25(3): 1443-1453.
6. Liu J, Zhang X, Chen x, Meng j (2011) On domination Number of Cartesian Product of directed paths. J Combinatorial Optimization 22(4): 651-662.
7. Shaheen R (2012) On the domination number of Cartesian products of two directed paths. Int J Contemp Math Sciences 7(36): 1785-1790.
8. Jacobson M, Kinch L (1983) On the domination number of products of graphs. Ars Combinatoria 18: 33-44.

DOI: 10.32474/CTBB.2020.02.000137